Essentials of Geometry

Postulates:

Ruler Postulate:
The points on a line can be matched one to one with the real number lines. The real number that corresponds to a point is the coordinate of the point. The distance between points A and B, written as AB, is the absolute value of the difference between the coordinates of A and B.

Segment Addition Postulate:
1. If B is between A and C,
   then AB + BC = AC.
2. If AB + BC = AC,
   then B is between A and C.

Protractor Postulate:
Consider $\overrightarrow{OB}$ and a point A on one side of $\overrightarrow{OB}$. The rays of the form $\overrightarrow{OA}$ can be matched one to one with the real numbers from 0 to 180. The measure of ∠AOB is equal to the absolute value of the difference between the real numbers for $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Angle Addition Postulate:
If P is in the interior of ∠RST,
then m∠RST = m∠RSP + m∠PST.

Reasoning and Proof

Postulates:

Through any two points there exists exactly one line.

A line contains at least two points.

If two lines intersect, then their intersection is exactly one point.

Through any three noncollinear points there exists exactly one plane.

A plane contains at least three noncollinear points.

If two points lie in a plane, then the line containing them lies in the plane.

If two planes intersect, then their intersection is a line.

Linear Pair Postulate:
If two angles form a linear pair, then they are supplementary.

Theorems:

Properties of Segment Congruence:
Segment congruence is reflexive, symmetric, and transitive.

Reflexive: For any segment AB, $\overline{AB} \cong \overline{AB}$.
Symmetric: If $\overline{AB} \cong \overline{CD}$,
then $\overline{CD} \cong \overline{AB}$.
Transitive: If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$,
then $\overline{AB} \cong \overline{EF}$.

Properties of Angles Congruence:
Angle congruence is reflexive, symmetric, and transitive.

Reflexive: For any angle A, $\angle A \cong \angle A$.
Symmetric: If $\angle A \cong \angle B$,
then $\angle B \cong \angle A$.
Transitive: If $\angle A \cong \angle B$ and $\angle B \cong \angle C$,
then $\angle A \cong \angle C$

Right Angles Congruence Theorem:
All right angles are congruent.

Congruent Supplements Theorem:
If two angles are supplementary to the same angle (or to congruent angles), then the two angles are congruent.

Congruent Complements Theorem:
If two angles are complementary to the same angle (or to congruent angles), then the two angles are congruent.

Vertical Angles Congruence Theorem:
Vertical Angles are congruent.

Parallel and Perpendicular Lines

Postulates:

Linear Pair Postulate
If two angles form a linear pair, then they are supplementary.

Parallel Postulate
If there is a line and a point not on the line, then there is exactly one line through the point parallel to the given line.

Perpendicular Postulate
If there is a line and a point not on the line, then there is exactly one line through the point perpendicular to the given line.

Corresponding Angles Postulate
If two parallel lines are cut by a transversal, then the pairs of corresponding angles are congruent.

Corresponding Angles Converse
If two lines are cut by a transversal so the corresponding angles are congruent, then the lines are parallel.

Slopes of Parallel Lines
In a coordinate plane, two non-vertical lines are parallel if and only if they have the same slope. Any two vertical lines are parallel.

Slopes of Perpendicular Lines
In a coordinate plane, two non-vertical lines are perpendicular if and only if the product of their slopes is –1. Horizontal lines are perpendicular to vertical lines.

THEOREMS:

Alternate Interior Angles Theorem
If two parallel lines are cut by a transversal, then the pairs of alternate interior angles are congruent.

Alternate Exterior Angles Theorem
If two parallel lines are cut by a transversal, then the pairs of alternate exterior angles are congruent.

Same-Side Interior Angles Theorem
If two parallel lines are cut by a transversal, then the pairs of same-side interior angles are supplementary.

Alternate Interior Angles Converse
If two lines are cut by a transversal so the alternate interior angles are congruent, then the lines are parallel.

Alternate Exterior Angles Converse
If two lines are cut by a transversal so the alternate exterior angles are congruent, then the lines are parallel.

Same-Side Interior Angles Converse
If two lines are cut by a transversal so the same-side interior angles are supplementary, then the lines are parallel.

Transitive Property of Parallel Lines
If two lines are parallel to the same line, then they are parallel to each other.

If two lines intersect to form a linear pair of congruent angles, then the lines are perpendicular.

If two lines are perpendicular, then they intersect to form four right angles.

If two sides of two adjacent acute angles are perpendicular, then the angles are complementary.

Perpendicular Transversal Theorem
If a transversal is perpendicular to one of two parallel lines, then it is perpendicular to the other.

Lines Perpendicular to a Transversal Theorem
In a plane, if two lines are perpendicular to the same line, then they are parallel to each other.

Congruent Triangles

Postulates:

Side-Side-Side (SSS) Congruence Postulate
If three sides of one triangle are congruent to three sides of a second triangle, then the two triangles are congruent.

Side-Angle-Side (SAS) Congruence Postulate
If two sides and the included angle of one triangle are congruent to two sides and the included angle of a second triangle, then the two triangles are congruent.

Angle-Side-Angle Congruence Postulate
If two angles and the included side of one triangle are congruent to two angles and the included side of a second triangle, then the two triangles are congruent.

THEOREMS:

Triangle Sum Theorem
The sum of the measures of the interior angles of a triangle is 180°

Corollary to the Triangle Sum Theorem
The acute angles of a right triangle are complementary.

Exterior Angle Theorem
The measure of an exterior angle of a triangle is equal to the sum of the measures of the two nonadjacent interior angles.

Third Angles Theorem
If two angles of one triangle are congruent to two angles of another triangle, then the third angles are also congruent.

Properties of Triangle Congruence
Triangle congruence is reflexive, symmetric, and transitive.

Reflexive: for any ΔABC, $\Delta ABC \cong \Delta ABC$

Symmetric: If $\Delta ABC \cong \Delta DEF$,
Then $\Delta DEF \cong \Delta ABC$

Transitive: If $\Delta ABC \cong \Delta DEF$ and $\Delta DEF \cong \Delta JKL$,
Then $\Delta ABC \cong \Delta JKL$,

Hypotenuse-Leg (HL) Congruence Theorem
If the hypotenuse and a leg of a right triangle are congruent to the hypotenuse and a leg of a second right triangle, then the two triangles are congruent.

Angle-Angle-Side (AAS) Congruence Theorem
If two angles and a non-included side of one triangle are congruent to two angles and the corresponding non-included side of a second triangle, then the two triangles are congruent.

Base Angles Theorem
If two sides of a triangle are congruent, then the angles opposite them are congruent.

Corollary to the Base Angles Theorem
If a triangle is equilateral, then it is equiangular.

Converse of the Base Angles Theorem
If two angles of a triangle are congruent, then the sides opposite them are congruent.

Corollary to the Converse of the Base Angles Theorem
If a triangle is equiangular, then it is equilateral.

Relationships within Triangles

Theorems:

Midsegment Theorem
The segment connecting the midpoints of two sides of a triangle is parallel to the third side and is half as long as that side.

Perpendicular Bisector Theorem
If a point is on a perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

Converse of the Perpendicular Bisector Theorem
If a point is equidistant from the endpoints of a segment, then it is on the perpendicular bisector of the segment.

Concurrency of Perpendicular Bisectors Theorem
The perpendicular bisectors of a triangle intersect at a point that is equidistant from the vertices of the triangle.

Angle Bisector Theorem
If a point is on the bisector of an angle, then it is equidistant from the two sides of the angle.

Converse of the Angle Bisector Theorem
If a point is in the interior of an angle and is equidistant from the sides of the angle, then it lies on the bisector of the angle.

Concurrency of Angle Bisectors of a Triangle
The angle bisectors of a triangle intersect at a point that is equidistant from the sides of the triangle.

Concurrency of Medians of a Triangle
The medians of a triangle intersect at a point that is two thirds of the distance from each vertex to the midpoint of the opposite side.

Concurrency of Altitudes of a Triangle
The lines containing the altitudes of a triangle are concurrent..

If one side of a triangle is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side.

If one angle of a triangle is larger than another angle, then the side opposite the larger angle is longer than the side opposite the smaller angle.

Triangle Inequality Theorem
The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Hinge Theorem
If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first is larger than the included angle of the second, then the third side of the first is longer than the third side of the second.

Converse of the Hinge Theorem
If two sides of one triangle are congruent to two sides of another triangle, and the third side of the first is longer than the third side of the second, then the included angle of the first is larger than the included angle of the second.

Similarity

Postulates:

Angle-Angle (AA) Similarity Postulate
If two angles of one triangle are congruent to two angles of another triangle, then the two triangles are similar.

Theorems:

Ratio of Perimeters of Similar Polygons
If two polygons are similar, then the ratio of their perimeters is equal to the ratios of their corresponding side lengths.

Side-Side-Side (SSS) Similarity Theorem
If the corresponding side lengths of two triangles are proportional, then the triangles are similar.

Side-Angle-Side (SAS) Similarity Theorem
If an angle of one triangle is congruent to an angle of a second triangle and the lengths of the sides including these angles are proportional, then the triangles are similar.

Triangle Proportionality Theorem
If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally.

Converse of the Triangle Proportionality Theorem
If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

If three parallel lines intersect two transversals, then they divide the transversals proportionally.

If a ray bisects an angle of a triangle, then it divides the opposite side into segments whose lengths are proportional to the lengths of the other two sides.

Right Triangles and Trigonometry

Theorems:

Pythagorean Theorem
In a right triangle, the square of the lengths of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

Converse of the Pythagorean Theorem
If the square of the length of the longest side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle.

Acute Triangle Theorem (Using Pythagorean Theorem)
If the square of the length of the longest side of a triangle is less than the sum of the squares of the lengths of the other two sides then the triangle is an acute triangle.

Obtuse Triangle Theorem (Using Pythagorean Theorem)
If the square of the length of the longest side of a triangle is greater than the sum of the squares of the lengths of the other two sides then the triangle is an obtuse triangle.

Altitude to the Hypotenuse Theorem
If the altitude is drawn to the hypotenuse of a right triangles, then the two triangles formed are similar to the original triangle and to each other.

Geometric Mean (Altitude) Theorem
In a right triangle, the altitude from the right angle to the hypotenuse divides the hypotenuse into two segments. The length of the altitude is the geometric mean of the lengths of the two segments.

Geometric Mean (Leg) Theorem
In a right triangle, the altitude from the right angle to the hypotenuse divides the hypotenuse into two segments. The length of each leg of the right triangle is the geometric mean of the lengths of hypotenuse and the segment of the hypotenuse that is adjacent to the leg.

45°-45°-90° Triangle Theorem
In a 45°-45°-90° triangle, the hypotenuse is $\sqrt{2}$ times as long as each leg.

30°-60°-90° Triangle Theorem
In a 30°-60°-90° triangle, the hypotenuse is twice as long as the shorter leg, and the longer leg is $\sqrt{3}$ times as long as the shorter leg.

SOH-CAH-TOA
Given ΔABC is a right triangle. Used to find the measure of a side.

$$\sin A = \frac{measure\ of\ opposite\ leg}{measure\ of\ the\ hypotenuse} \qquad \cos A = \frac{measure\ of\ adjacent\ leg}{measure\ of\ the\ hypotenuse} \qquad \tan A = \frac{measure\ of\ the\ opposite\ leg}{measure\ of\ the\ adjacent\ leg}$$

Inverse Trigonometric Functions

$$sin^{-1}\left(\frac{measure\ of\ opposite\ leg}{measure\ of\ the\ hypotenuse}\right) = m\angle A;\quad cos^{-1}\left(\frac{measure\ of\ adjacent\ leg}{measure\ of\ the\ hypotenuse}\right) = m\angle A;\quad tan^{-1}\left(\frac{measure\ of\ opposite\ leg}{measure\ of\ the\ adjacent\ leg}\right) = m\angle A$$

Law of Sines
Given ΔABC with side lengths a, b, and c.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Law of Cosines
Given ΔABC with side lengths a, b, and c.

$a^2 = b^2 + c^2 - 2bc \cdot cosA$
$b^2 = a^2 + c^2 - 2ac \cdot cosB$
$c^2 = a^2 + b^2 - 2ab \cdot cosC$

Quadrilaterals

Theorems:

Polygon Interior Angles Theorem
The sum of the measures of the interior angles of a convex n-gon = $180(n-2)°$

Polygon Exterior Angles Theorem
The sum of the measures of the exterior angles of a convex polygon, one angle at each vertex, is 360°

Parallelogram Theorems

A) If a quadrilateral is a parallelogram,
Then its opposite sides are congruent.

B) If a quadrilateral is a parallelogram,
Then its opposite angles are congruent.

C) If a quadrilateral is a parallelogram,
Then its consecutive angles are supplementary.

D) If a quadrilateral is a parallelogram,
Then its diagonals bisect each other.

E) If both pairs of opposite sides of a quadrilateral are congruent,
Then the quadrilateral is a parallelogram.

F) If both pairs of opposite angles of a quadrilateral are congruent,
Then the quadrilateral is a parallelogram.

G) If one pair of opposite sides of a quadrilateral are congruent and parallel,
Then the quadrilateral is a parallelogram.

H) If the diagonals of a quadrilateral bisect each other,
Then the quadrilateral is a parallelogram.

Rhombus Corollary
A quadrilateral is a rhombus if and only if it has four congruent sides.

Rectangle Corollary
A quadrilateral is a rectangle if and only if it has four right angles.

Square Corollary
A quadrilateral is a square if and only if it is a rhombus and a rectangle.

Rhombus Theorems

A) A parallelogram is a rhombus if and only if its diagonals are perpendicular.

B) A parallelogram is a rhombus if and only if each diagonal bisects a pair of opposite angles.

Rectangle Theorem
A parallelogram is a rectangle if and only if its diagonals are congruent.

Trapezoid Theorems

A) If a trapezoid is isosceles,
Then both pairs of base angles are congruent.

B) If a trapezoid has a pair of congruent base angles,
Then it is an isosceles trapezoid.

C) A trapezoid is isosceles if and only if its diagonals are congruent.

Midsegment Theorem for Trapezoids
The Midsegment of a trapezoid is parallel to each base and its length is one half the sum of the lengths of the bases.

Kite Theorems

A) If a quadrilateral is a kite,
Then its diagonals are perpendicular.

B) If a quadrilateral is a kite,
Then exactly one pair of opposite angles are congruent.

Properties of Circles

Postulates:

Arc Addition Postulate
The measure of an arc formed by two adjacent arcs is the sum of the measure of the two arcs.

Theorems:

Perpendicular Tangent Theorem
In a plane, a line is tangent to a circle if and only if the line is perpendicular to a radius of the circle at its endpoint on the circle.

Tangent Intersection Theorem
Tangent segments from a common external point are congruent.

In the same circle, or in congruent circles, two minor arcs are congruent if and only if their corresponding chords are congruent.

If one chord is a perpendicular bisector of another chord,
then the first chord is a diameter.

If a diameter of a circle is perpendicular to a chord,
Then the diameter bisects the chord and its arc.

In the same circle, or in congruent circles, two chords are congruent if and only if they are equidistant from the center.

Measure of an Inscribed Angle Theorem
The measure of an inscribed angle is one half the measure of its intercepted arc.

If two inscribed angles of a circle intercept the same arc,
Then the angles are congruent.

If a right triangle is inscribed in a circle,
Then the hypotenuse is a diameter of the circle.
Conversely, if one side of an inscribed triangle is a diameter of the circle,
Then the triangle is a right triangle and the angle opposite the diameter is the right angle.

A quadrilateral can be inscribed in a circle if and only if its opposite angles are supplementary.

If a tangent and a chord intersect at a point on a circle,
Then the measure of each angle formed is one half the measure of its intercepted arc

Angles Inside the Circle
If two chords intersect inside a circle,
Then the measure of each angle is one half the sum of the measures of the arcs intercepted by the angle and its vertical angle.

Angles Outside the Circle
If a tangent and a secant, two tangents, or two secants intersect outside a circle,
Then the measure of the angle formed is one half the difference of the measures of the intercepted arcs.

Segments of Chords Theorem
If two chords intersect in the interior of a circle,
Then the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other chord.

Segments of Secants Theorem
If two secant segments share the same endpoint outside a circle,
Then the product of the lengths of one secant segment and its external segment equals the product of the lengths of the other secant segment and its external segment.

Segments of Secants and Tangents Theorem
If a secant segment and a tangent segment share an endpoint outside of a circle,
Then the product of the lengths of the secant segment and its external segment equals the square of the length of the tangent segment.